

Spectral Triples on Proper Etale Groupoids

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Abstract

A proper etale Lie groupoid is modelled as a (noncommutative) spectral geometric space. The spectral triple is built on the algebra of smooth functions on the groupoid base which are invariant under the groupoid action. Stiefel-Whitney classes in Lie groupoid cohomology are introduced to measure the orientability of the tangent bundle and the obstruction to lift the tangent bundle to a spinor bundle. In the case of an orientable and spin Lie groupoid, an invariant spinor bundle and an invariant Dirac operator will be constructed. This data gives rise to a spectral triple. The algebraic orientability axiom in noncommutative geometry is reformulated to make it compatible with the geometric model.

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Introduction

The goal of this work is to take a step towards a unification of Lie groupoid theory and (noncommutative) spectral geometry. We will restrict the study to the proper etale Lie groupoids. The classical orbifolds are represented by a Lie groupoid with these properties [10], [9]. Roughly speaking, orbifold theory is a geometric model for manifolds subject to local smooth actions of finite group [14]. The essential feature in this formalism is that one can have several different groups acting on different regions of the manifolds. The Lie groupoid theory puts the orbifold theory to a more general geometric context. On the other hand, one can understand Lie groupoids as "local charts" of differentiable stacks. The dictionary between these two theories is provided in [1]. In this work, the relationship to yet another geometric theory, spectral geometry, will be studied. The approach in the spectral geometry is to put the geometric structures in operator theoretic framework: in the case of compact manifolds, one can recover the whole riemannian geometry from such data, [5]. This theory is often referred to noncommutative geometry since it allows to study noncommutative algebras as well, [4]. In fact, some interesting noncommutative deformations of function algebras on groupoids have been developed recently, see for example [2]. However, the geometric study of these algebras (in terms of spectral triples) still lacks the classical limit.

The spectral geometric model is based on a topological algebra. In this work, the algebras of interest are the subalgebras of smooth functions on the base of

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the groupoids which are invariant under the action of the groupoid arrows. One also needs a Dirac operator and a complex Hilbert space of spinors. The role of the Dirac operator is to measure the metric properties. The spinor space has a natural action by the groupoid arrows which is inherited from the action of the groupoid on the tangent bundle by the local diffeomorphism associated to the arrows. The spectral triple model will be based on the invariance under the action.

The approach in this work builds on a localization of a proper étale Lie groupoid. This is a groupoid theoretic analogue for fixing a good cover for a manifold on which all the bundles get a locally trivial structure and the cohomology can be computed from the combinatorics of the good cover. If the base of a Lie groupoid is given a cover, then one can decompose the manifolds of arrows to components consisting of the arrows between these cover sheets. This defines a localization of the whole groupoid structure. An analogue of a good cover is a nice refinement: this is a localization in which all the components and their intersections in the base and in the arrow manifolds can be contracted to a finite number of points. If this localization exists, then one can compute the groupoid cohomology from it by applying the simplicial structure of the localized Lie groupoid. Due to Morita invariance of cohomology, this is independent of the choice of a nice refinement. This strategy has been applied elsewhere in the literature, for example in the theory of gerbes and twisted K-theory on orbifold groupoids, [8], [15].

In the differential geometric context, one can build spinor bundles as follows. The tangent bundle τX on a manifold X is equipped with a riemannian structure which gives a reduction for the structure group of τX to the orthogonal group O_n . Provided that τX is orientable, the structure group reduces further to SO_n . The obstruction for this reduction is measured by nontriviality of the first Stiefel-Whitney class $w_1(\tau X) \in H^1(X, \mathbb{Z}_2)$. The bundle structure and the Stiefel-Whitney class are convenient to code in a Čech cohomology data associated to a fixed good cover. There is a covering homomorphism $\varpi : \text{Spin}_n \rightarrow SO_n$ and if τX is orientable, one can try to lift the transition cocycle of the bundle τX through ϖ . The obstruction for this lift is measured by nontriviality of the second Stiefel-Whitney class $w_2(\tau X) \in H^2(X, \mathbb{Z}_2)$ which is convenient to realize in the Čech cohomology as well. If the obstruction vanishes one gets a spinor bundle by a reconstruction from the transition cocycle. In addition, one gets a Dirac operator acting on the spinors and after a Hilbert space completion, there is a classical spectral triple if X is geodesically complete.

In the case of a proper étale Lie groupoid $\Theta \rightrightarrows X$, there is a tangent bundle τX on the base manifold X . It is naturally equipped with an action of the groupoid Θ by the local diffeomorphisms associated to the arrows Θ . We shall proceed by localizing the groupoid and writing down a transition cocycle for τX which determines a class in the groupoid cohomology. The cocycle contains the information of the topological structure of τX and of the Θ -action. The bundle τX can be equipped with a riemannian structure for which the groupoid acts orthogonally. This leads to an O_n reduction of the transition cocycle. The SO_n reduction is associated to the orientability of the groupoid. A Stiefel-Whitney class is introduced to measure the obstruction to be able to do the reduction. Then Spin_n lifting of the SO_n valued cocycles will be studied: this leads to the spin structures and to the second Stiefel-Whitney class in the groupoid cohomology. A spinor bundle with a groupoid action is then reconstructed from

the transition cocycles. The sections in this spinor bundle can be completed with a natural L^2 -inner product. The Hilbert space has a Θ -invariant subspace. The Dirac operator in this model is also Θ -invariant and therefore acts on the invariant subspace. This will lead to a spectral triple for the algebra $C^\infty(X)^\Theta$ - the Θ -invariant subalgebra of smooth functions.

There is an approach to define spinors on Lie groupoids which applies the theory of Lie algebroids, [12]. More specifically, the tangent bundle in this formalism is defined to be the kernel subbundle for the tangent map dt of the target morphism t pulled back to the base by the unit morphism in the Lie groupoid. In the case under consideration t is a local diffeomorphism and therefore the tangent spaces would be the zero vector spaces. Therefore this approach would not be useful here.

The structures of spectral triples associated to orbifolds have been studied in [13]. However, in this reference the consideration has been restricted to the global action orbifolds which are manifolds subject to a global action by a finite group. It is pointed out in [13] that in this special case the spectral triple does not satisfy the algebraic orientability axiom for a spectral triple [5]. We shall reformulate the algebraic orientation so that the new orientation is a consequence of the geometric orientation which will be formulated in terms of Lie groupoid technology.

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Notation. We use the notation X_\bullet to denote a Lie groupoid with a base manifold $X_{(0)}$ and a manifold of k -times composable arrows $X_{(k)}$ for all $k \in \mathbb{N}$. We also write occasionally $X_{(1)} = \Theta$ and $X_{(0)} = X$ and then use the symbol $\Theta \rightrightarrows X$ to denote the groupoid. Θ and X are smooth manifolds which are both Hausdorff and second countable. The target and source maps: $s, t : \Theta \rightarrow X$ are smooth submersions. X_\bullet has a simplicial structure. The face maps $\partial_k^i : X_{(k)} \rightarrow X_{(k-1)}$ are defined by

$$\partial_k^i(\sigma_k, \dots, \sigma_1) = \begin{cases} (\sigma_{k-1}, \dots, \sigma_1), & i = 1 \\ (\sigma_k, \dots, \sigma_{i+1}, \sigma_i \sigma_{i-1}, \sigma_{i-2}, \dots, \sigma_1), & i \neq 1, k \\ (\sigma_k, \dots, \sigma_2), & i = n \end{cases} \quad (1)$$

and the degeneracy maps $s_k^i : X_{(k)} \rightarrow X_{(k+1)}$ are defined by

$$s_k^i(\sigma_k, \dots, \sigma_1) = (\sigma_k, \dots, \sigma_i, \mathbf{1}_{s(\sigma_i)}, \sigma_{i-1}, \dots, \sigma_1).$$

$\mathbf{1}_x$ denotes the unit morphism at $x \in X$ on the base. If U and V are subsets in X we define the following subspaces in Θ :

$$\Theta_U = s^{-1}(U), \quad \Theta^V = t^{-1}(V) \quad \text{and} \quad \Theta_U^V = s^{-1}(U) \cap t^{-1}(V)$$

If $x \in X$ and $U = V = \{x\}$ then Θ_x^x is the stabilizer group at x . The orbit space (coarse moduli space) of X_\bullet is denoted by $|X_\bullet|$.

All groupoids are Lie groupoids in this work. The parameter n will denote the dimension of the base manifold X everywhere below.

1 Local Structure

1.1. A groupoid X_\bullet is proper if $(s, t) : \Theta \rightarrow X \times X$ is a proper map and etale if s and t are local diffeomorphisms. Both, the arrow and the base manifolds are

assumed to be locally compact and second countable. Suppose that $\{N_a : 1 \leq a \leq k\}$ is a collection of open subsets in the base of the Lie groupoid $\Theta \rightrightarrows X$. Then we adopt the following notation for the restriction of the $(k-1)$ -times composable arrows to these open sets

$$\Theta(N_k \cdots N_1) := \Theta_{N_{k-1}}^{N_k} s \times_t \cdots s \times_t \Theta_{N_2}^{N_3} s \times_t \Theta_{N_1}^{N_2}.$$

The symbol $s \times_t$ is used for the fibre products.

A cover of a manifold is said to be acyclic with respect to a Čech resolution if an arbitrary intersection of its cover sheets consists of a disjoint union of open subsets which are all contractible to a point. It is shown in [10] (Theorem 4.1, 4 \Rightarrow 1) that given a proper étale groupoid X_\bullet , then at any point $x \in X$ there is an open nbd U such that X_\bullet localizes to a transformation groupoid $\Theta_x^x \times U \rightrightarrows U$. In fact, the nbd U can be taken to be a euclidean ball on which the stabilizer acts as a subgroup of the orthogonal group. Thus, proper étale groupoids can be given local charts, exactly as in the case of an orbifold. Although, we do not require that these local charts would satisfy the local embedding property of orbifolds. The analysis of [11] (Corollary 1.2.5) gives us the following localization result:

Proposition. Let X_\bullet be a proper étale groupoid. Then there exists a numerable cover $\{N_a : a \in I\}$ of X so that the following hold:

- (1) The collection $\{N_a : a \in I\}$ is a good open cover of X .
- (2) For all $k \in \mathbb{N}$, the collection

$$\{\Theta(N_{a_{k+1}} \cdots N_{a_1}) : a_1, \dots, a_{k+1} \in I\}$$

is an open cover of the manifold of k -times composable arrows $X_{(k)}$, and this cover is acyclic with respect to the Čech resolution.

1.2. A refinement of a proper étale groupoid $\Theta \rightrightarrows X$ is a proper étale groupoid $\Xi \rightrightarrows Y$ together with an étale groupoid morphism $\phi : Y_\bullet \rightarrow X_\bullet$ inducing a Morita equivalence, i.e. $\phi_0 : Y \rightarrow X$ is an étale map which induces a surjection $|Y_\bullet| \rightarrow |X_\bullet|$ between the orbit spaces and the following diagram is cartesian:

$$\begin{array}{ccc} \Xi & \longrightarrow & Y \times Y \\ \downarrow & & \downarrow \downarrow \\ \Theta & \longrightarrow & X \times X \end{array}$$

where the horizontal maps are (s, t) and the vertical maps ϕ_1 and $\phi_0 \times \phi_0$.

Consider a proper étale groupoid X_\bullet with an open cover $\{N_a\}$ of X . Then we can define the refinement

$$\coprod_{ab} \Theta_{N_b}^{N_a} \rightrightarrows \coprod_a N_a. \quad (2)$$

We refer this as the Čech groupoid \tilde{X}_\bullet associated to the choice of the cover. If the cover $\{N_a\}$ has the properties (1) and (2) in the Proposition 1.1, then the Čech groupoid is called a nice refinement of X_\bullet . Since the open subsets N_a can be chosen to be arbitrarily small, we can assume that each N_a is equipped with its own coordinate functions: $\varphi_a : N_a \rightarrow \mathbb{R}^n$.

1.3. A vector bundle over a Lie groupoid X_\bullet is a smooth vector bundle with a typical fibre V on the base which is equipped with a Θ -action

$$\begin{array}{ccc}
& \searrow & \xi \\
\Theta & \xRightarrow{\quad} & X \\
& \downarrow \pi & \\
& & X
\end{array}$$

The domain of the action is the fibre product $\Theta_s \times_\pi \xi$ and if $\sigma \in \Theta_x$ is an arrow and u is a vector in the fibre ξ_x over $x \in X$, then σ acts by

$$(\sigma, u_x) \mapsto (\rho(\sigma)u)_{\sigma x}$$

so that u_x maps to the fibre $\xi_{\sigma x}$ over $\sigma x \in X$ and $\rho : \Theta \rightarrow GL(V)$ is required to satisfy

$$\rho(\tau)\rho(\sigma) = \rho(\tau\sigma), \quad \rho(\mathbf{1}_x) = \iota$$

for all $(\tau, \sigma) \in X_{(2)}$ and unit arrows $\mathbf{1}_x$. Here the vector space V is always taken to be of finite rank but one can take it to be real or complex.

An inner product in a vector bundle ξ over the groupoid X_\bullet is a smoothly varying inner product in the fibres of ξ which is invariant under the action of Θ .

Proposition. A vector bundle over a proper etale groupoid can be equipped with an inner product.

Proof. We can use the existence of an inner product in the bundle ξ together with an averaging trick over the s -fibres of X_\bullet which provides the Θ -invariance. For this we shall need a Haar system and a cutoff function. A Haar system is a collection of measures $\mu = \{\mu_x\}_{x \in X}$ which are supported in $\Theta_x = s^{-1}(x)$ such that:

- (1) For all $f \in C_c^\infty(\Theta)$, $x \mapsto \int_{\sigma \in \Theta_x} f(\sigma) \mu_x(d\sigma)$ is smooth function on X .
- (2) The measures are invariant under the right translations by the arrows:

$$\int_{\sigma \in \Theta_{t(\sigma')}} f(\sigma\sigma') \mu_{t(\sigma')}(d\sigma) = \int_{\sigma \in \Theta_{s(\sigma')}} f(\sigma) \mu_{s(\sigma')}(d\sigma).$$

A function $c : X \rightarrow \mathbb{R}_+$ is called a cutoff if the integration of $c \circ t$ over each s -fibre on X satisfies

$$\int_{\sigma \in \Theta_x} c(t(\sigma)) \mu_x(d\sigma) = 1$$

and for all compact sets $K \subset X$, the support of $(c \circ t)|_{\Theta_K}$ is compact. A Haar measure and a cutoff always exists in a proper Lie groupoid.

Let (\cdot, \cdot) denote an inner product in the bundle ξ . Then we choose a cutoff function c and a Haar system μ and define

$$(v, w)_x^I = \int_{\tau \in \Theta_x} c(t(\tau)) (\rho(\tau)v, \rho(\tau)w)_{t(\tau)} \mu_x(d\tau).$$

The linearity and the conjugate symmetry follow from these properties in (\cdot, \cdot) and from the linearity of the integration. For all $x \in X$

$$(v, v)_x^I = \int_{\tau \in \Theta_x} c(t(\tau)) \|\rho(\tau)v\|_{t(\tau)}^2 \mu_x(d\tau)$$

is positive since c is a positive function, the norm is positive and the Haar measure maps positive functions to positive reals. Moreover, if $(v, v)_x^I = 0$ then the function $c(t(\tau))\|\rho(\tau)v\|_{t(\tau)}^2$ needs to be zero for all $\tau \in \Theta_x$. So the positive definiteness follows from the positive definiteness of (\cdot, \cdot) and from the positivity of c .

If $\sigma \in \Theta_x$ is an arrow, then

$$\begin{aligned} (\rho(\sigma)v, \rho(\sigma)w)_{t(\sigma)}^I &= \int_{\tau \in \Theta_{t(\sigma)}} c(t(\tau))(\rho(\tau)\rho(\sigma)v, \rho(\tau)\rho(\sigma)w)_{t(\tau)} \mu_{t(\sigma)}(d\tau) \\ &= \int_{\tau \in \Theta_{t(\sigma)}} c(t(\tau\sigma))(\rho(\tau)\rho(\sigma)v, \rho(\tau)\rho(\sigma)w)_{t(\tau\sigma)} \mu_{t(\sigma)}(d\tau) \\ &= \int_{\tau \in \Theta_{t(\sigma)}} c(t(\tau\sigma))(\rho(\tau\sigma)v, \rho(\tau\sigma)w)_{t(\tau\sigma)} \mu_{t(\sigma)}(d\tau) \\ &= \int_{\tau \in \Theta_{s(\sigma)}} c(t(\tau))(\rho(\tau)v, \rho(\tau)w)_{t(\tau)} \mu_x(d\tau) \\ &= (v, w)_x^I. \end{aligned}$$

The fourth equality applies the defining property of the Haar measure on X_\bullet .

□

1.4. For each arrow $\sigma \in \Theta_x$ there is an open nbd V of x and a local section of s , $\hat{\sigma} : V \rightarrow \Theta$, such that $\hat{\sigma}(x) = \sigma$ and $t \circ \hat{\sigma}$ is an open embedding. In the case of an etale Lie groupoid, any two such sections agree on their common domain. Now the assignment

$$\sigma \mapsto \varphi_\sigma = t \circ \hat{\sigma} : V \rightarrow \varphi_\sigma(V).$$

defines a local diffeomorphisms. Denote by $\Delta(\Theta)$ the set of germs of local bisections.

Let τX denote the tangent bundle over the base X . We can apply the elements of $\Delta(\Theta)$ to define a Θ -action on τX . For any arrow $\sigma : x \rightarrow y$ there is a germ of local diffeomorphisms φ_σ which amounts to define the differential map $(d\varphi_\sigma)_x$. This gives the action

$$\Theta_s \times_\pi \tau X \ni (\sigma, [x, v]) \mapsto [\sigma x, (d\varphi_\sigma)_x(v)] \in \tau X,$$

where we have applied a local trivialization of τX around x in the standard way. This is indeed well defined since for a composable pair $(\tau, \sigma) \in X_{(2)}$ the differentials satisfy $(d\tau)_{t(\sigma)}(d\sigma)_{s(\sigma)} = d(\tau \circ \sigma)_{s(\sigma)}$.

The Proposition 1.3 provides a Θ -invariant real inner product for the bundle τX . A groupoid equipped with an inner product in τX will be referred to a riemannian groupoid.

A riemannian groupoid is defined to be oriented if for all elements in the Lie groupoid $\Delta(\Theta)$, the Jacobian matrix associated to the linear transformation

$$(d\varphi_\sigma)_x : \tau X_x \rightarrow \tau X_{\sigma x}$$

is an element in the group of invertible linear transformations with positive determinant GL_n^+ . Notice that this implies that τX is an oriented vector bundle in the ordinary sense: at each $x \in X$ there is the identity morphism $\mathbf{1}_x$ and

the local bisection associated to the unit arrows are identity maps. Especially the differentials of the unit maps are simply the Jacobians associated to the coordinate transformations and in the case of an orientable groupoid it has a GL_n^+ reduction implying that τX is an orientable in the ordinary sense.

1.5. The goal in the following is to describe the tangent bundle and the orientability condition in terms of cohomological data. For this we shall apply the sheaf cohomology theory on Lie groupoids, [1]. The general principle is to work with a double complex in which one of the directions is determined by a cocycle complex arising from the simplicial structure of the arrows and another direction is determined by an injective resolution. This theory is invariant under the Morita equivalence. In the case of a Lie groupoid, the injective resolutions can be replaced by Čech resolutions, up to an isomorphism. The strategy here is to replace the proper étale groupoid X_\bullet with a Čech groupoid \check{X}_\bullet which gives a nice refinement for X_\bullet : the cover in the Čech-construction has the properties (1,2) in the proposition 1.1. The cohomology groups remain invariant since the localization preserves the Morita equivalence class. The Čech direction in the double complex does not contribute to the cohomology now, and it is sufficient to compute the simplicial cohomology in the Čech groupoid. Moreover, any two localizations of a proper étale groupoid are Morita equivalent with each other and therefore the cohomology is independent on this choice.

For all $k > 1$ we have the $k + 1$ arrows

$$\partial_k^i : X_{(k)} \rightarrow X_{(k-1)}$$

associated to the simplicial structure, (1). Then we set

$$\partial_k = \sum_{i=1}^{k+1} (-1)^{i-1} \partial_k^i.$$

In the case $k = 1$ we set $\partial = t - s$. These maps satisfy $\partial^2 = 0$. Let \mathcal{G} denote a sheaf of smooth functions valued in an abelian group G . Then dually, we can set a cochain complex

$$\bigoplus_{k \geq 0} C^\infty(X_{(k)}, \mathcal{G})$$

with a coboundary operator $\partial^* : C^\infty(X_{(k)}, \mathcal{G}) \rightarrow C^\infty(X_{(k+1)}, \mathcal{G})$. The cohomology of this complex will be called the simplicial cohomology of X_\bullet with values in the sheaf \mathcal{G} . To compute the Lie groupoid cohomology of a proper étale groupoid X_\bullet we can proceed by choosing a cover for which \check{X}_\bullet determines a nice refinement, and then apply the simplicial cohomology construction.

As usual, one can also define the order one cohomology groups with values in a sheaf of smooth maps getting values in a nonabelian group.

Let X_\bullet be a proper étale groupoid and \check{X}_\bullet a Čech groupoid associated to an open cover $\{N_a : a \in I\}$ of X which gives a nice refinement. The tangent bundle τX is subject to the action of the groupoid Θ as discussed in 1.4. This action determines the following structure cocycle

$$g \in \prod_{ab} \Theta_{N_b}^{N_a} \rightarrow GL_n, \quad g(\sigma) = (d\varphi_\sigma)_{s(\sigma)}$$

for all $\sigma \in \Theta_{N_b}^{N_a}$ where the differential is computed with respect to the coordinate charts in N_b and N_a . When written out, the cocycle condition is just the compatibility of the action with the composition of arrows. Thus, g represents a class in $H^1(\check{X}_\bullet, \mathcal{GL}_n)$. Recall that we can assume that each N_a is equipped with its own coordinate system. The components of g arising from the unit arrows of X_\bullet have a special importance. If $\mathbf{1}_p \in \Theta_{N_a}^{N_b}$ this means that $p \in N_a \cap N_b \neq \emptyset$ and then

$$g(\mathbf{1}_p) = (d\iota)_p = J(p)_a^b$$

is just the usual Jacobian matrix computed from the change of coordinate charts.

Proposition 1. Let X_\bullet be a riemannian proper etale groupoid and \check{X}_\bullet a Cech groupoid defining a nice refinement of X_\bullet with respect to the open good cover $\{N_a\}$ of the base. Then the structure cocycle of τX is valued in O_n and if τX is orientable the structure cocycle has an SO_n reduction.

Proof. The target group of the structure cocycle can be reduced to O_n because the Θ action is orthogonal with respect to the riemannian structure. In the orientable case all the matrix elements have strictly positive determinant and so they are in SO_n . \square

In the case of a proper etale action groupoid the orientability implies that the group acts by orientation-preserving isometries.

One can also reconstruct bundles from the cocycle data. In this process both, the topological structure and the Θ -action can be recovered. Moreover, only the cohomology class of the cocycle is important.

Proposition 2. Let X_\bullet be the groupoid of Proposition 1. The tangent bundle τX is fully determined by its structure cocycle g in the cohomology group $H^1(\check{X}_\bullet, O_n)$. Any element in the cohomology class of g produces a vector bundle that is isomorphic to τX as a vector bundle over X_\bullet .

An isomorphism of vector bundles over a Lie groupoid is an isomorphism of smooth vector bundles so that the isomorphism commutes with the Θ actions on both bundles.

Proof. A smooth vector bundle can be reconstructed from the cocycle data in the standard way. Define the total space of τX by

$$\tau X = \left[\coprod_a N_a \times \mathbb{R}^n \right] / \sim$$

where \sim is the equivalence relation which is determined by the restriction of the cocycle g to the unit arrows of X_\bullet : if $\mathbf{1}_p \in \Theta_{N_b}^{N_a}$ then $g(\mathbf{1}_p) = (d\iota_p)$ and

$$N_b \times \mathbb{R}^n \ni (p, v) \sim (p, (d\iota_p)_b^a v) \in N_a \times \mathbb{R}^n.$$

The trivialization ϑ_{N_a} over N_a sends a pair $(p, v) \in N_a \times \mathbb{R}^k$ to its equivalence class $[p, v]$ in the bundle. The Θ -action can be reconstructed by setting

$$\sigma \cdot [p, v] = [\sigma p, (d\varphi_\sigma)v],$$

for any $\sigma \in \Theta_{N_b}^{N_a}$, $p \in N_b$ and $\sigma(p) \in N_a$. Now the unit arrows of X_\bullet act as identities.

Any cocycle that is cohomologous to g is of the form

$$g' \in \prod_{ab} \Theta_{N_b}^{N_a} \rightarrow O_n \quad \text{with the local components}$$

$$g'(\sigma) = (f_{N_a} \circ t)(\sigma)g(\sigma)(f_{N_b} \circ s)^{-1}(\sigma)$$

for any $\sigma \in \Theta_{N_b}^{N_a}$ and some smooth functions $f_{N_b} : N_b \rightarrow O_n$ and $f_{N_a} : N_a \rightarrow O_n$. When restricted to the unit arrows of X_\bullet , the cocycle g' defines a smooth vector bundle which is isomorphic to τX and which is trivialized by

$$\vartheta'_{N_a}(p, v) = \vartheta_{N_a}(p, f_{N_a}(p)v)$$

over N_a . This isomorphism respects the Θ -action. Suppose that $\sigma \in \Theta_{N_b}^{N_a}$ is the arrow $p \mapsto \sigma p$, then

$$\begin{aligned} \sigma \cdot \vartheta'_{N_b}(p, v) &= \vartheta'_{N_b}(\sigma p, g'(\sigma)v) \\ &= \vartheta_{N_b}(\sigma p, f_{N_a}(\sigma p)g(\sigma)v) \\ &= \vartheta'_{N_a}(\sigma p, g(\sigma)v). \end{aligned}$$

and so the bundle isomorphism which changes the trivialization commutes with the action. \square

Consider the exact group extension sequence $1 \rightarrow SO_n \rightarrow O_n \rightarrow \mathbb{Z}_2 \rightarrow 0$ in which the second map is the inclusion $i : SO_n \hookrightarrow O_n$. This induces a long exact sequence in the groupoid sheaf cohomology

$$\cdots \rightarrow H^0(\check{X}_\bullet, \mathbb{Z}_2) \rightarrow H^1(\check{X}_\bullet, SO_n) \rightarrow H^1(\check{X}_\bullet, O_n) \rightarrow H^1(\check{X}_\bullet, \mathbb{Z}_2) \rightarrow \cdots$$

The image of the structure cocycle g of τX under the map in cohomology $q_* : H^1(\check{X}_\bullet, O_n) \rightarrow H^1(\check{X}_\bullet, \mathbb{Z}_2)$ which is induced by the quotient map is called the first Stiefel-Whitney class of X_\bullet ,

$$q_*([g]) = w_1(X_\bullet) \in H^1(\check{X}_\bullet, \mathbb{Z}_2).$$

The following result follows from a simple diagram chase.

Corollary. Suppose that X_\bullet is a proper etale groupoid. The class $w_1(X_\bullet)$ is the obstruction class for the orientability of X_\bullet .

2 Spinor Bundles

In this section we assume that X_\bullet is an orientable proper etale groupoid and \check{X}_\bullet is a Čech groupoid which defines a nice refinement of X_\bullet with respect to the open good cover $\{N_a\}$ of X .

2.1. Let $\text{cl}(n)$ denote a real Clifford algebra generated by an n -dimensional vector space with a euclidean inner product. The spin group Spin_n is a subgroup in the group of invertibles in $\text{cl}(n)$, see [7]. There is a covering homomorphism $\varpi : \text{Spin}_n \rightarrow SO_n$. We have $\text{Spin}_1 = O_1$ and Spin_2 is isomorphic to SO_2 . In the latter case ϖ is a 2-fold covering homomorphism. For $n > 2$, ϖ is a universal covering map with kernel equal to \mathbb{Z}_2 . The complex spinor module (ρ_s, Σ) is an

irreducible complex representation for the complexification of $\text{cl}(n)$. The group Spin_n acts on the spinor module through its embedding to the Clifford algebra and this gives the representation

$$\rho_s : \text{Spin}_n \rightarrow GL(\Sigma).$$

The Clifford algebra is a Spin_n -module algebra under the adjoint action of Spin_n . More precisely, if $\gamma : \mathbb{R}^n \rightarrow \text{cl}(n) \otimes \mathbb{C}$ is the canonical embedding, then for all $a \in \text{Spin}_n$ and $\gamma(u) \in \text{cl}(n) \otimes \mathbb{C}$ we get a group action

$$\text{Ad}(a)\gamma(u) = a\gamma(u)a^{-1} = \gamma(\rho(\varpi(a))u)$$

where $\rho : SO_n \rightarrow SO(\mathbb{R}^n)$ denotes the representation by matrix multiplication.

The cotangent spaces τX_x^* equipped with the dual riemannian structure r_x^{-1} determine n -dimensional Clifford algebras at each $x \in X$.

In the following we shall assume that $n \geq 2$. Consider the bundle τX determined by the Lie groupoid cocycle g with a class in $H^1(\check{X}_\bullet, SO_n)$. The cocycle g can be pulled back through the covering morphism $\varpi : \text{Spin}_n \rightarrow SO_n$ which results a cochain

$$\hat{g} := \varpi^*(g) \in \prod_{ab} \Theta_{N_b}^{N_a} \rightarrow \text{Spin}_n.$$

This set of locally defined functions can be applied to define a bundle of Clifford algebras over X_\bullet . Under the adjoint action of Spin_n on the Clifford algebra, the center vanishes implying the relations

$$\text{Ad}(\hat{g}(\tau))\text{Ad}(\hat{g}(\sigma)) = \text{Ad}(\hat{g}(\tau\sigma))$$

for all composable pairs $(\tau, \sigma) \in \check{X}_{(2)}$. In fact, this makes $\text{Ad}(\hat{g})$ a cocycle in the cohomology group $H^1(\check{X}_\bullet, SO(\text{cl}(n)))$. Then the reconstruction determines a bundle of Clifford algebras which is trivialized over the cover $\{N_a\}$ and in which the Θ -action is defined by

$$\sigma \cdot [p, e_p] = [\sigma p, \text{Ad}(\varpi^*(d\varphi_\sigma)_p)(e_p)].$$

for $\sigma : p \rightarrow \sigma p$. Let us denote by $\text{CL}(X_\bullet)$ the Clifford bundle.

In the space of sections the Θ -action induces a pullback action by

$$(\sigma \cdot e)_p = \text{Ad}(\varpi^*(d\varphi_\sigma)_p)^{-1}(e_{\sigma p}); \quad \sigma : p \rightarrow \sigma p.$$

Given two sections of the Clifford bundle one can apply the Clifford multiplication fiberwise to define a product in the space of sections. Since the Θ -action is the inverse adjoint action the multiplication and the Θ -action commute with each other. This makes $\text{Sec}(\text{CL}(X_\bullet))$ a Θ -module algebra.

2.2. The cochain \hat{g} is not necessarily a cocycle since the kernel of the covering morphism ϖ is the subgroup \mathbb{Z}_2 . It thus follows that although g is a cocycle its lift through ϖ does not need to be. The lifting problem is cohomological in nature. Associated to the lift there is the short exact sequence of groups

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_n \rightarrow SO_n \rightarrow 1$$

This group extension sequence induces a long exact sequence in the Lie groupoid cohomology in the usual sense [1]

$$\cdots \rightarrow H^1(\check{X}_\bullet, \mathbb{Z}_2) \rightarrow H^1(\check{X}_\bullet, \text{Spin}_n) \rightarrow H^1(\check{X}_\bullet, \text{SO}_n) \rightarrow H^2(\check{X}_\bullet, \mathbb{Z}_2) \rightarrow \cdots$$

The connecting homomorphism $\delta_* : H^1(\check{X}_\bullet, \text{SO}_n) \rightarrow H^2(\check{X}_\bullet, \mathbb{Z}_2)$ is the map which applies the groupoid cohomology coboundary operator to the lifted cochain: $\delta_*(g) = \partial^*(\hat{g})$. Therefore the nontriviality of this class is an obstruction to lift g to a groupoid cocycle and therefore an obstruction to define a spinor bundle with a compatible Θ -action by the reconstruction from \hat{g} . Following the usual terminology we call

$$\delta_*(g) := w_2(X_\bullet) \in H^2(\check{X}_\bullet, \mathbb{Z}_2)$$

the second Stiefel-Whitney class of the proper etale groupoid X_\bullet . If this class is nontrivial, one gets a groupoid central extension which are classified by $H^2(\check{X}_\bullet, \mathbb{T})$, [16]. In this case, \mathbb{Z}_2 is viewed as a subgroup in \mathbb{T} . The proper etale groupoid is called spin if $w_2(X_\bullet)$ is trivial. The nontrivial cases are groupoid spin gerbes, although they will not be discussed here.

Proposition. Let X_\bullet be an orientable proper etale groupoid. The tangent bundle τX can be lifted to the spinor bundle F_Σ if and only if the second Stiefel-Whitney class $w_2(X_\bullet) \in H^2(\check{X}_\bullet, \mathbb{Z}_2)$ is zero.

Finally, if $n = 1$ then $\text{cl}(1) \otimes \mathbb{C} = \mathbb{C}$ and its irreducible complex representation is one dimensional. Since X_\bullet is orientable, the structure cocycle of τX gets values in SO_1 which is the trivial group. Therefore the spinor bundle in this case is just the trivial complex line bundle over X . Now the Θ action simply translates the fibres.

3 Spectral Triple

3.1. Consider a vector bundle ξ over X_\bullet . Let $\Theta_s \times_\pi \xi \rightarrow \xi$ be a left action of the arrows Θ on ξ . The action restricts to a linear isomorphisms between the fibres and induces a smooth map $\Theta_s \times_\iota X \rightarrow X$ on the base. For an arrow $\sigma \in \Theta_x$ we have a local bisection $\hat{\sigma}$ defined locally in a nbd V_x of x and the associated diffeomorphism φ_σ . Given a local section ψ of ξ defined in $\varphi_\sigma(V_x)$ we can pull it back to V_x by

$$\varphi_\sigma^\#(\psi)_p = \rho^{-1}(\hat{\sigma}(p))(\psi \circ \varphi_\sigma(p))$$

where ρ is associated to the groupoid action, as in 1.3. This can be extended to the map

$$\begin{aligned} \varphi_\sigma^\# : \Lambda^k(X, \xi)|_{\varphi_\sigma(V_x)} &\rightarrow \Lambda^k(X, \xi)|_{V_x}; \\ (\varphi_\sigma^\# \Phi)(p, v_1, \dots, v_k) &= \rho^{-1}(\hat{\sigma}(p))\Phi(\varphi_\sigma(p), (d\varphi_p)v_1, \dots, (d\varphi_p)v_k) \end{aligned}$$

where $\Lambda^*(X, \xi)$ is the space of ξ -valued differential forms on X and v_i are tangent vectors at $p \in V_x$.

A complex valued smooth function $f \in C^\infty(X)$ is Θ -invariant if $f(x) = f(y)$ holds whenever there is an arrow $\sigma : x \rightarrow y$. This is equivalent to the invariance

of f under the local pullbacks $\varphi_\sigma^\#$ applied with the trivial representation ρ on the complex line. Denote by $C^\infty(X)^\Theta$ the space of Θ -invariant functions. A section ψ in ξ is called Θ -invariant if it is invariant under the local pullbacks by the local diffeomorphisms $\Delta(\Theta)$.

Consider a vector bundle ξ over X_\bullet . Then we call a connection ∇ in ξ a geometric connection if it is a connection in the differential geometric sense: ∇ is a linear map $\nabla : \text{Sec}(\xi) \rightarrow \Lambda^1(X, \xi)$ which satisfies the Leibnitz rule

$$\nabla(f\psi) = df \otimes \psi + f\nabla(\psi)$$

for all $f \in C^\infty(X)$ and $\psi \in \text{Sec}(\xi)$. A groupoid connection in ξ is a geometric connection which is invariant under the pullbacks by the local diffeomorphisms $\Delta(\Theta)$:

$$\varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} = \nabla_x$$

for all $\sigma \in \Theta_x$.

Although geometric connections always exist and they form an affine space over the space of sections of the homomorphism bundle $\text{Hom}(\xi, \xi \otimes \tau X^*)$ it is not obvious that one can always find one with the Θ -invariance property. In the case under consideration they do exist.

Proposition If ξ is a smooth vector bundle over a proper etale groupoid X_\bullet . Then there is a groupoid connection.

Proof. Choose a cutoff $c : X \rightarrow \mathbb{R}_+$ and a Haar system in X_\bullet (recall the Proposition 1.3). Let ∇ be any geometric connection in ξ . Then define

$$\nabla_x^I = \int_{\sigma \in \Theta_x} c(t(\sigma)) \varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} \mu_x(d\sigma). \quad (3)$$

The function $c \circ t$ has a finite support over each s -fibre Θ_x and so the integration with respect to the Haar measure reduces to a finite sum fiberwise. For each $x \in X$ and $\sigma \in \Theta_x$ the composition $\varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1}$ maps a section of ξ at $x \in X$ to a section of $\Lambda(X, \xi)$ at $x \in X$. By the linearity of the space of sections, ∇_x^I is a map

$$\nabla_x^I : \text{Sec}(\xi)_x \rightarrow \Lambda(X, \xi)_x$$

Since the Haar integration is a smooth operation, the assignment $x \mapsto \nabla_x^I \psi_x$ defines a smoothly varying section in $\Lambda(X, \xi)$ for all $\psi \in \text{Sec}(\xi)$.

The linearity of ∇^I is obvious. For the Leibnitz rule we write locally $\nabla_{\sigma x} = d_{\sigma x} + A_{\sigma x}$ for all $\sigma \in \Theta_x$. Then

$$\begin{aligned} \varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} &= \rho(\hat{\sigma})^{-1} \varphi_\sigma^* (d_{\sigma x} + A_{\sigma x}) (\varphi_\sigma^{-1})^* \rho(\hat{\sigma}) \\ &= d_x + \rho(\hat{\sigma})^{-1} (d\rho(\hat{\sigma}))_x + \rho(\hat{\sigma})^{-1} \varphi_\sigma^* A_{\sigma x} (\varphi_\sigma^{-1})^* \rho(\hat{\sigma}) \\ &= d_x + \rho(\hat{\sigma})^{-1} (d\rho(\hat{\sigma}))_x + \text{Ad}(\rho(\hat{\sigma}))^{-1} A_x. \end{aligned}$$

The exterior differential term is constant in the direction of the fibres of s and therefore the connection ∇^I is of the form $d + A^I$ with $A^I \in \Lambda^1(X) \otimes \mathfrak{gl}_n$. A linear map $\text{Sec}(\xi) \rightarrow \Lambda(X, \xi)$ of this form satisfies the Leibnitz rule.

For the invariance under $\Delta(\Theta)$, take $\tau \in \Theta$ be the arrow $x' \rightarrow x$. Then

$$\begin{aligned}
\varphi_\tau^\# \nabla_{\tau x}^I (\varphi_\tau^\#)^{-1} &= \int_{\sigma \in \Theta_x} c(t(\sigma)) \varphi_\tau^\# \varphi_\sigma^\# \nabla_{\sigma \tau x} (\varphi_\tau^\# \varphi_\sigma^\#)^{-1} \mu_x(d\sigma) \\
&= \int_{\sigma \in \Theta_x} c(t(\sigma \tau)) \varphi_{\sigma \tau}^\# \nabla_{\sigma \tau x} (\varphi_{\sigma \tau}^\#)^{-1} \mu_x(d\sigma) \\
&= \int_{\sigma \in \Theta_{x'}} c(t(\tau)) \varphi_\tau^\# \nabla_{\tau x} (\varphi_\tau^\#)^{-1} \mu_x(d\tau) \\
&= \nabla_x^I. \quad \square
\end{aligned}$$

Suppose that ξ has an inner product so that Θ acts unitarily (orthogonally in the real case). The groupoid connection ∇ is called unitary (riemannian in the real case), if it compatible with the inner product in the usual sense:

$$d(\psi_1, \psi_2) = (\nabla \psi_1, \psi_2) + (\psi_1, \nabla \psi_2)$$

for all smooth sections ψ_1, ψ_2 of ξ .

Corollary. If ξ is a smooth vector bundle over a proper etale groupoid X_\bullet . Then there is a unitary (riemannian) groupoid connection.

Proof. Given any inner product in ξ there is always a geometric connection which satisfies the compatibility condition. When written out in local coordinates, this means that the connection coefficients are 1-forms with values in the Lie algebra \mathfrak{u}_n (\mathfrak{o}_n in the real case). The construction in the proposition now provides a unitary (riemannian) groupoid connection. \square

3.2. Now we proceed towards the main goal. Suppose that X_\bullet is an orientable spin proper etale groupoid. Suppose that τX is equipped with a fixed Θ -invariant inner product. Associated to the inner product we have the Clifford bundle $\text{CL}(\tau X^*)$ and the spinor bundle F_Σ over X_\bullet . The sections of the Clifford bundle act on the sections of the spinor bundle. Let us denote by $\text{Sec}(\text{CL}(\tau X^*))^\Theta$ and $\text{Sec}(F_\Sigma)^\Theta$ the spaces of smooth sections that are invariant under the action of $\Delta(\Theta)$. Since the action on the former is by adjugation, it is obvious that the fiberwise Clifford multiplication determines a module structure:

$$\text{Sec}(\text{CL}(\tau X))^\Theta \times \text{Sec}(F_\Sigma)^\Theta \rightarrow \text{Sec}(F_\Sigma)^\Theta.$$

A groupoid connection in $\text{Sec}(\text{CL}(\tau X^*))$ clearly restricts to a linear map

$$\nabla_{\text{CL}} : \text{Sec}(\text{CL}(\tau X))^\Theta \rightarrow \Lambda(X, \text{CL}(\tau X))^\Theta.$$

The same holds for the groupoid connection ∇ in F_Σ . If in addition ∇_{CL} and ∇ are compatible with the module structure, then ∇ is called Clifford compatible.

Proposition. Given an orientable spin proper etale groupoid X_\bullet , there exists a Clifford compatible unitary groupoid connection in the spinor bundle F_Σ .

Proof. We proceed by choosing geometric connections ∇_{CL} and ∇ in $\text{CL}(\tau X^*)$ and in F_Σ which are Clifford compatible and ∇ is a unitary connection. Then we follow the Proposition 3.1 and define the invariant connections ∇_{CL}^I and ∇^I .

The unitarity follows automatically by the Corollary in 3.1. For all $\sigma \in \Theta_x$, $e \in \text{Sec}(\text{CL}(\tau X^*))$ and $\psi \in \text{Sec}(F_\Sigma)$ we have

$$\begin{aligned}\varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} e \psi &= \varphi_\sigma^\# \nabla_{\sigma x} ((\text{Ad}(\varphi_\sigma^\#)^{-1} e) ((\varphi_\sigma^\#)^{-1} \psi)) \\ &= \varphi_\sigma^\# ((\nabla_{\text{CL}})_{\sigma x} (\text{Ad}(\varphi_\sigma^\#)^{-1} e)) ((\varphi_\sigma^\#)^{-1} \psi) \\ &+ \varphi_\sigma^\# ((\text{Ad}(\varphi_\sigma^\#)^{-1} e) \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} \psi) \\ &= (\text{Ad}(\varphi_\sigma^\#) (\nabla_{\text{CL}})_{\sigma x} \text{Ad}(\varphi_\sigma^\#)^{-1} (e)) \psi_x \\ &+ e_x (\varphi_\sigma^\# \nabla_{\sigma x} (\varphi_\sigma^\#)^{-1} \psi),\end{aligned}$$

where we have written Ad for the adjoint action of $\Delta(\Theta)$ on the Clifford sections. Then

$$\nabla^I(e\psi) = (\nabla_{\text{CL}}^I e)\psi + e\nabla\psi$$

follows from the formula (3). \square

3.3. Let us fix a groupoid riemannian structure in the bundle τX over X_\bullet . Locally we can choose n linearly independent vector fields e_i of τX with the dual vector fields e_i^* with respect to the riemannian structure. Associated to the riemannian structure we have Clifford and spinor bundles over the groupoid. Let ∇ be a Clifford compatible unitary groupoid connection in F_Σ . The Dirac operator is a differential operator $\tilde{\partial} : \text{Sec}(F_\Sigma) \rightarrow \text{Sec}(F_\Sigma)$ given by

$$\tilde{\partial} = \sum_{i=1}^n \gamma(e_i^*) \nabla_{e_i}.$$

From the point of view of the base manifold X (as an ordinary manifold), $\tilde{\partial}$ is an ordinary Dirac operator acting on the space of smooth spinor fields as usual. Fundamental is the following.

Proposition 1. The Dirac operator is invariant under the Θ -action. In particular, it can be restricted to $\text{Sec}(F_\Sigma)^\Theta$.

Proof. If u is a vector field in τX , then the invariance of ∇ implies

$$\varphi_\sigma^\# \nabla_{u_{\sigma x}} (\varphi_\sigma^\#)^{-1} = \nabla_{(d\varphi_\sigma)^{-1} u_x}.$$

for any $\sigma \in \Theta_x$. It follows

$$\begin{aligned}\varphi_\sigma^\# \tilde{\partial}_{\sigma x} (\varphi_\sigma^\#)^{-1} &= (\text{Ad}(\varphi_\sigma^\#) (\gamma(e_i^*)))_{\sigma x} \varphi_\sigma^\# \nabla_{(e_i)_{\sigma x}} (\varphi_\sigma^\#)^{-1} \\ &= \text{Ad}(\rho(\hat{\sigma})^{-1}) \gamma(e_i^*)_x \nabla_{(d\varphi_\sigma)^{-1} (e_i)_x} \\ &= \gamma(((d\varphi_\sigma)^{-1})^* e_i^*)_x \nabla_{(d\varphi_\sigma)^{-1} (e_i)_x} \\ &= \tilde{\partial}_x.\end{aligned}$$

The last step holds since the module structures in τX^* and τX are dual to each other and e_i and e_i^* are dual basis. \square

To make $\tilde{\partial}$ a formally self-adjoint operator we need to set one more condition for the Clifford structure. Namely, the unit covector fields $\gamma(u)$ in $\text{Sec}(\text{CL}(\tau X^*))$ act on the spinors unitarily

$$(\gamma(u)\psi_1, \gamma(u)\psi_2) = (\psi_1, \psi_2).$$

A Θ -invariant inner product which satisfies this conditions exists. To see this it is sufficient to choose an inner product with this property and then average out the Θ action as in section 3.1. A complex spinor bundle is called a complex Dirac bundle on X_\bullet if it is equipped with a unitary Θ -invariant groupoid connection which is Clifford compatible, and a Θ -invariant inner product which is normalized so that the units of $\text{Sec}(\text{CL}(\tau X^*))$ act as unitary transformations.

Let $\text{Sec}(F_\Sigma)$ denote the space of smooth sections in the spinor bundle and let $(\cdot, \cdot) : \text{Sec}(F_\Sigma) \times \text{Sec}(F_\Sigma) \rightarrow C^\infty(X)$ be the natural pairing in the fibres.

Proposition 2. A complex Dirac bundle F_Σ exists on an orientable spin proper etale groupoid. The Dirac operator satisfies

$$(\psi_1, \tilde{\partial}\psi_2) - (\tilde{\partial}\psi_1, \psi_2) = \text{div}(V) \quad (4)$$

where $\psi_i \in \text{Sec}(F_\Sigma)^\Theta$ and V is a vector field determined by $\Phi(V) = (\psi_1, \gamma(\Phi)\psi_2)$ for all $\Phi \in \Lambda^1(X)$.

Proof. The existence of a complex Dirac bundle is proved above. The second part can be proved as in the usual geometric case, [7] II.5.3.

Notice that the divergence on the right side of (4) is Θ invariant (because the left side is).

3.4. The Dirac operator $\tilde{\partial}$ fits in the definition of a spectral triple only if we can complete the space of spinor sections in such a way that $\tilde{\partial}$ is a formally self-adjoint: this requires that the divergences vanish under integration. We shall also concentrate on a compact case, that is, X_\bullet is called compact if the orbit space $|X_\bullet|$ is compact. Compactness is independent on the Morita equivalence class since Morita equivalence leaves $|X_\bullet|$ invariant.

Recall that a proper etale groupoid can be given an open cover $\{N_a\}$ of X so that X_\bullet localizes as a transformation groupoid $\Theta_x^x \ltimes N_a \rightrightarrows N_a$ where $x \in N_a$ and the stabilizer acts through the local diffeomorphisms. Denote by $\Pi : X \rightarrow |X_\bullet|$ the projection. Under these assumptions, one can construct a triangulation of the orbit space $|X_\bullet|$ with the properties (for a proof see [10] Lemma 1.2.2):

- (1) The singular locus of $|X_\bullet|$ lies in a subcomplex of the triangulation.
- (2) The triangulation refines the open cover $\Pi(N_a)$ of $|X_\bullet|$.
- (3) The triangularization of $|X_\bullet|$ lifts to a triangularization of X : for every simplex S on $|X_\bullet|$ there is a simplex \tilde{S} in X such that

$$\Pi|_{\tilde{S}} := \tilde{S} \longrightarrow S$$

is a homeomorphism.

Suppose that X_\bullet is orientable. Given a triangulation satisfying (1-3) we can define an integration of invariant functions in X over the orbit space $|X_\bullet|$. Suppose that S is a simplex in $|X_\bullet|$. Then $S \in \Pi(N_a)$ for some a . Then one defines an integration of a Θ -invariant function f in X over the simplex S in $|X_\bullet|$ by

$$\int_S f = \int_{\tilde{S}} f d\mu_a$$

where the right side is the riemannian integration on N_a over the lifted simplex. Notice that the lifted simplex is not uniquely defined, however, given the different lifts \tilde{S}_1 and \tilde{S}_2 , these map to each other under the local diffeomorphisms which are orientation preserving isometries. The integration is extended over $|X_\bullet|$ by

$$\int_{|X_\bullet|} f = \sum_a \int_{S_a} f$$

where a runs over the index set of the triangulation of $|X_\bullet|$. Suppose that the triangulation of $|X_\bullet|$ has no boundary: each face of a simplex is a face of exactly one other simplex. Then we also say that X_\bullet has no boundary. In this case, the divergences vanish under integration: see Section 1 of [3], for instance.

Let us define an inner product in $\text{Sec}(F_\Sigma)$ by

$$\langle \psi_1, \psi_2 \rangle = \int_{|X_\bullet|} (\psi_1, \psi_2). \quad (5)$$

Then Proposition 2 of 3.4 gives:

Proposition. If X_\bullet is compact with an empty boundary, then the Dirac operator acting on the complex Dirac bundle is formally self-adjoint.

3.5. A spectral triple for the algebra $C^\infty(X)^\Theta$ is data $(C^\infty(X)^\Theta, \mathcal{H}, \bar{D})$ where \mathcal{H} is a Hilbert space on which $C^\infty(X)^\Theta$ has a bounded involutive $*$ -representation and the Dirac operator \bar{D} acts as a densely defined self adjoint operator such that $[\bar{D}, f]$ is a bounded operator for all $f \in C^\infty(X)^\Theta$. Whenever X is even dimensional the spectral triple is required to have a chiral grading, namely a bounded operator ω with

$$\omega^2 = 1, \quad \{\omega, \gamma(e^*)\} = \{\omega, \bar{D}\} = [\omega, f] = 0 \quad (6)$$

for all $e^* \in \tau X^*$ and $f \in C^\infty(X)^\Theta$. Moreover, one says that the spectral triple is finitely summable if $(1 + \bar{D}^2)^{-m}$ is a trace class operator for some finite $m \in \mathbb{N}$.

The complex Hilbert space can be constructed by a completion of the space of Θ -invariant sections $\text{Sec}(F_\Sigma)^\Theta$ with respect to the L^2 -inner product (5). Let us denote by $L^2(F_\Sigma)^\Theta$ this completion. Suppose that the Dirac operator \bar{D} has an extension on $L^2(F_\Sigma)^\Theta$ which is still denoted by \bar{D} . The Clifford module has a canonical section defined by

$$\omega = i^{\frac{n}{2}} \gamma(e_1^*) \cdots \gamma(e_n^*)$$

if $e_i^* : i \in \{1, \dots, n\}$ define an orthonormal basis fiberwise. This satisfies (6) whenever n (the dimension of X and Θ) is even. If n is odd, then $\omega = 1$. Moreover, this section transforms as a volume form, and since the adjoint action of Θ acts on the covectors as orientation preserving rotations (X_\bullet is orientable and the inner product is Θ -invariant), this section is Θ -invariant. Therefore ω acts on $L^2(F_\Sigma)^\Theta$.

Theorem. Let X_\bullet be a compact orientable spin proper etale groupoid with an empty boundary and let F_Σ be a Dirac bundle on X_\bullet . Then

$$(C^\infty(X)^\Theta, \bar{D}^\Theta, L^2(F_\Sigma)^\Theta)$$

defines a finitely summable spectral triple, and if X is even dimensional, then the spectral triple is equipped with a chirality operator ω .

Proof. It is obvious that the algebra $C^\infty(X)^\Theta$ has an action on $\text{Sec}(F_\Sigma)^\Theta$ by pointwise multiplication. This extends to a bounded $*$ -representation on $L^2(F_\Sigma)^\Theta$. The $*$ -structure is clearly Θ -invariant and the representation is faithful. The standard proof of self-adjointness of the closure of the Dirac operator ([6] Theorem 9.15) can be applied in this setup straightforwardly: the proof only requires self-adjointness in a dense domain, which in our case is $\text{Sec}(F_\Sigma)^\Theta$, and the property that \mathfrak{D} is a pseudodifferential operator of order 1 on X , which holds since \mathfrak{D} is a classical Dirac operator. Similarly arguing we see that $[\mathfrak{D}, f]$ has to be bounded on $L^2(F_\Sigma)^\Theta$ since the commutators $[\mathfrak{D}, f]$ are given exactly as in the classical case, [7] II.5.5. The finite summability of \mathfrak{D} holds since $\text{Sec}(F_\Sigma)^\Theta$ is a subspace in the space of all sections of the spinor bundle F_Σ and the Dirac spectrum satisfies the finite summability in the latter case since X is a finite dimensional manifold. \square

3.6. The orientability of the spectral triple on $C^\infty(X)^\Theta$ is equivalent to the existence of a Hochschild cycle

$$c = \sum_{i=1}^k a_i^0 \otimes a_i^1 \otimes \cdots \otimes a_i^n \in Z_n(C^\infty(X)^\Theta, C^\infty(X)^\Theta)$$

such that when represented on $L^2(F_\Sigma)^\Theta$

$$\sum_{i=1}^k a_i^0 [\mathfrak{D}, a_i^1] \cdots [\mathfrak{D}, a_i^n] = \omega. \quad (7)$$

It is proved in [13] that in the case of a global action orbifold, the orientation cannot be satisfied unless the group action on X is free. This is because of the lack of invariant functions to construct the Hochschild orientation cycle. The orientation holds in the following weaker form: there exists a Hochschild cycle

$$c = \sum_{i=1}^k a_i^0 \otimes a_i^1 \otimes \cdots \otimes a_i^n \in Z_n(C^\infty(X), C^\infty(X))$$

such that (7) holds and ω defines a bounded Θ -invariant operator on $L^2(F_\Sigma)^\Theta$. As argued above, if we have a spectral triple on a proper etale Lie groupoid, then the geometric orientation of X_\bullet defined in 1.4 implies the algebraic weakened orientation condition.

In the case of a compact manifold X , one can identify the differential n -forms with the Hochschild n -homology classes of the algebra $C^\infty(X)$. The reason why the usual spectral triple orientation fails in the study of invariant smooth functions is that, unless the groupoid acts freely, one cannot identify the invariant n -forms with the Hochschild n -homology classes of $C^\infty(X)^\Theta$. For example in the case of an oriented proper etale Lie groupoid one has transition functions of a tangent bundle which get values in SO_n . Then there is a nowhere vanishing volume form

$$\sum_{i=1}^k a_i^0 da_i^1 \wedge \cdots \wedge da_i^n$$

which is invariant since it transforms fiberwise as a determinant of the transition cocycle. Therefore we have an invariant form but according to [13] one cannot have $a_i \in C^\infty(X)^\Theta$ for all the indices i . One needs to modify the Hochschild-Kostant-Rosenberg-Connes theorem to have a proper algebraic correspondence for the invariant forms.

3.7. The theorem of 3.4 is a step towards a unification of the Lie groupoid theory and spectral geometry. Given a spectral triple one can apply the tools in noncommutative geometry in the study of Lie groupoids. For example, one gets a noncommutative differential calculus with an integration theory which can be used in the study of homotopy invariants, such as cyclic cohomology and K-theory of the Θ -invariant algebra. In addition to the orientability, there are secondary axioms for a spectral triple of a commutative algebra. In the case of a compact smooth manifolds one can recover the riemannian manifold out of the full data. Moreover, these extra structures make a spectral triple more applicable. Since the construction of the spectral triple of 3.4 proceeds by passing to a Θ -invariant Hilbert subspace, it is an easy matter to check that the order one condition and regularity, [5], hold in the case of a spectral triple over a proper etale groupoid.

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